Math 520

Volume: Disc Method

§6.2

This section shows you how to calculate the volume of a solid object if you can determine its cross sectional areas. The types of solids include irregular ones. To find the volume, we will set up an integral using an expression for the area of a typical cross section.

> **Volume by Slicing**
>
> Suppose the cross-sectional area of a solid cut by planes perpendicular to an $x-$axis is known to be $A(x)$ for each $x \in [a, b]$. The volume of a typical slice is approximately $A(x_i^*)\Delta x$ and thus the total volume obtained by this method of slicing is
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> $$V = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i^*)\Delta x = \int_a^b A(x)\, dx.$$
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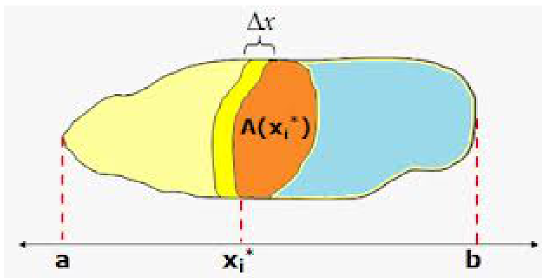
> 
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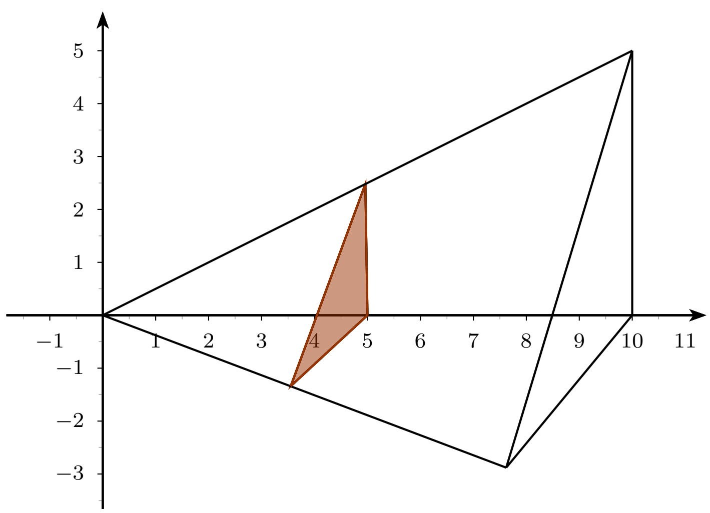
1. A crystal prism is 10cm long. Its cross sections are right triangles whose heights are formed by the line $y = \frac{1}{2}x$ and whose bases are twice the height. Find the volume of the prism.



> **Solution:** $A(x) = \frac{1}{2}bh = \frac{1}{2}\left(\frac{1}{2}x\right)x = \frac{1}{4}x^4$. So the volume is
>
> $$V = \int_a^b A(x)\, dx = \int_0^{10} \frac{1}{4}x^2 = \left.\frac{x^3}{12}\right|_0^{10} = \frac{1000}{12}.$$

A solid of revolution is an object obtained by rotating planar region about a line. Usually this line is the $x-$axis, $y-$axis, or some other line. Sometimes the resulting figure will have a crosection in the shape of a disk. The volume will be approximated by a sum of volumes of disks which suggest the definite integral.

Volume by Disc Method

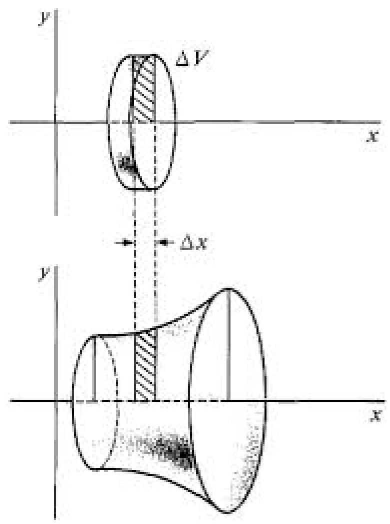
If the region under $y = f(x)$ is rotated about the $x-$axis, the volume may be approximated by a sum of volumes of disks.

The volume of a typical disk is

$$\pi(\text{radius})^2(\text{thickness}) = \pi[f(x_1^*)]^2\Delta x.$$

Thus the volume is

$$V = \int_a^b \pi[f(x)]^2\, dx.$$



2. Determine the volume of the solid of revolution obtained by rotating $y = 4 - x^2$, $y = 0$, $x = 0$, and $x = 2$ about the $x-$axis.

Solution:

Volume of the disk= $\pi(4 - (x_i^*)^2)^2\Delta x$.

Volume of solid is= $\displaystyle\int_0^2 \pi(4 - x^2)^2\, dx = \frac{256}{15}\pi.$

3. Determine the volume of the solid of revolution obtained by rotating the region bounded by $y = x^2$, the $x-$axis, and the vertical lines $x = 1$ and $x = 2$ about the $x-$axis.

Solution:

Volume of the disk= $\pi((x_i^*)^2)^2\Delta x$.

Volume of solid is= $\displaystyle\int_1^2 \pi(x)^4)\, dx = \frac{31}{5}\pi.$

4. Determine the volume of the solid of revolution obtained by rotating the region bounded by $y = -x^2 + 2x$, the $x-$axis, and the vertical lines $x = 0$ and $x = 2$ about the $x-$axis.

Solution:

Volume of the disk= $\pi(-x^2 + 2x)^2\Delta x$.

Volume of solid is= $\displaystyle\int_0^2 \pi(-x^2 + 2x)^2\, dx = \frac{16}{15}\pi.$

5. Determine the volume of the solid of revolution obtained by rotating the region bounded by $y = \sqrt{4 - x^2}$, the $x-$axis, and the vertical lines $x = 0$ and $x = 2$ about the $x-$axis.

Solution:

Volume of the disk= $\pi(\sqrt{4 - x^2})^2 \Delta x$.

Volume of solid is= $\int_0^2 \pi(\sqrt{4 - x^2})^2 \, dx = \frac{32}{3}\pi$.