Math 520

U-Substitution with Definite Integrals

§5.5

This section gives a technique for evaluating definite integrals that involve the composition of functions. It is the Chain Rule stated in terms of definite integrals.
You have two options:

1. Carry out the method of substitution all the way on the corresponding indefinite integral, then compute the difference of the values $x = b$ and $x = a$ as usual.

2. After making the substitutions for $u = g(x)$ and $du = g'(x)\ dx$, also substitute for the limits of integration: $u = g(a)$ when $x = a$ and $u = g(b)$ when $x = b$

> A Formula for Integration by Substitution
>
> If f, u, and g are continuous functions such that
>
> $$f(x) = g'(u(x)) \cdot u'(x),$$
>
> then
>
> $$\int_a^b f(x)\ dx = \int_{u(a)}^{u(b)} g(u)\ du = G(u)\Big|_{u(a)}^{u(b)} = G(u(b)) - G(u(a))$$
>
> or
>
> $$\int_a^b f(x)\ dx = \int_{x=a}^{x=b} g(u)\ du = G(u)\Big|_{x=a}^{x=b} = G(u(x))\Big|_a^b = G(u(b)) - G(u(a)).$$

Example: $\displaystyle\int_0^1 2x(x^2+1)^3\ dx$

Solution: Let $\boxed{u = x^2 + 1}$. Then $\frac{du}{dx} = 2x$ and $\boxed{du = 2x\ dx}$.
Putting the boxed equations together our integral has changed into something we can integrate.
Option 1:

$$\begin{aligned}\int_0^1 2x(x^2+1)^3\ dx &= \int_1^2 u^3\ du \\ &= \frac{u^4}{4}\Big|_1^2 \\ &= \frac{2^4}{4} - \frac{1^4}{4} \\ &= \frac{16}{4} - \frac{1}{4} = \frac{15}{4}\end{aligned}$$

Option 2:

$$\begin{aligned}\int_0^1 2x(x^2+1)^3\ dx &= \int_{x=0}^{x=1} u^3\ du \\ &= \frac{u^4}{4}\Big|_{x=0}^{x=1} \\ &= \frac{(x^2+1)^4}{4}\Big|_0^1 \\ &= \frac{(1^2+1)^4}{4} - \frac{(0^2+1)^4}{4} \\ &= \frac{16}{4} - \frac{1}{4} = \frac{15}{4}\end{aligned}$$

1. $\displaystyle\int_0^1 \sqrt{2x+1}\, dx$

Solution: $\sqrt{3} - \dfrac{1}{3}$

2. $\displaystyle\int_{-\pi}^{\pi} \sin x \cos x \, dx$

Solution: 0

3. $\displaystyle\int_{-1}^{1} 2x(x^2+1)^4 \, dx$

Solution: 0

4. $\displaystyle\int_0^1 \frac{x^5}{(x^6+1)^3} \, dx$

Solution: $\frac{1}{16}$

5. $\displaystyle\int_{-2}^{2} x\sqrt{x+2} \, dx$

Solution: $\frac{32}{15}$

6. $\displaystyle\int_1^3 \frac{3}{3x+1} \, dx$

Solution: $\ln 10 - \ln 4$

7. $\displaystyle\int_{-1}^{2} 6xe^{3x^2} \, dx$

Solution: $e^3 - e^{12}$