Math 520

U-Substitution Part 2

§5.5

This section gives a technique for evaluating indefinite integrals that involve the composition of functions. It is the Chain Rule stated in terms of indefinite integrals.

> **A Formula for Integration by Substitution**
>
> If f and u are functions such that $f'(u(x)) \cdot u'(x)$ is integrable, then
>
> $$\int f'(u(x)) \cdot u'(x)\, dx = f(u(x)) + C.$$
>
> Your choice of the function u is often inside a composition and whose derivative appears elsewhere.

Sometimes you will need to introduce a constant so that the formula is balanced.

Example: $\int x(x^2+1)^3\, dx$

Solution: Let $\boxed{u = x^2+1}$. Then $\frac{du}{dx} = 2x$ and $du = 2x\, dx$, but this doesn't match our integrand, we need $x\, dx$. So if we multiply both sides by $\frac{1}{2}$ we get $\boxed{\frac{1}{2}\, du = x\, dx}$.

Putting the boxed equations together our integral has changed into something we can integrate.

$$\begin{aligned}\int x(x^2+1)^3\, dx &= \int \frac{1}{2}u^3\, du \\ &= \frac{1}{2}\int u^3\, du \\ &= \frac{1}{2}\cdot\frac{u^4}{4} + C \\ &= \frac{(x^2+1)^4}{8} + C\end{aligned}$$

Example: $\int x(x-3)^6\, dx$

Solution: Let $\boxed{u = x-3}$. Then $\frac{du}{dx} = 1$ and $\boxed{du = dx}$, but this doesn't match our integrand, we need $x\, dx$. Since $u = x-3$ we can solve this for x and get $\boxed{u+3 = x}$. Putting the boxed equations together our integral has changed into something we can integrate.

$$\begin{aligned}\int x(x-3)^6\, dx &= \int (u+3)u^6\, du \\ &= \int (u^7 + 3u^6)\, du \\ &= \frac{u^8}{8} + 3\frac{u^7}{7} + C \\ &= \frac{(x-3)^8}{8} + 3\frac{(x-3)^7}{7} + C\end{aligned}$$

1. $\int x^5(x^6+13)^4\, dx$

> **Solution:** $\frac{1}{30}(x^6+13)^5 + C$

2. $\displaystyle\int \frac{1}{3x+1}\,dx$

Solution: $\frac{1}{3}\ln(3x+1)+C$

3. $\displaystyle\int x^2\sqrt{1+x^3}\,dx$

Solution: $\frac{2}{9}(x^3+1)^{3/2}+C$

4. $\displaystyle\int 6x\sin(x^2)\,dx$

Solution: $-3\cos x^2+C$

5. $\displaystyle\int x^6 e^{5x^7-2}\,dx$

Solution: $\dfrac{1}{35}e^{5x^2-2}+C$

6. $\displaystyle\int x(x+4)^5\,dx$

Solution: $\dfrac{(x+4)^7}{7}-4\dfrac{(x+4)^6}{6}+C$

7. $\displaystyle\int x\sqrt{x+1}\,dx$

Solution: Let $\boxed{u=x+1}$. Then $\frac{du}{dx}=1$ and $\boxed{du=dx}$, but this doesn't match our integrand, we need $x\,dx$. Since $u=x+1$ we can solve this for x and get $\boxed{u-1=x}$. Putting the boxed equations together our integral has changed into something we can integrate.

$$\begin{aligned}\int x\sqrt{x+1}\,dx &= \int (u-1)\sqrt{u}\,du\\ &= \int (u^{3/2}+u^1/2)\,du\\ &= \frac{2}{5}u^{5/2}+\frac{2}{3}u^{3/2}+C\\ &= \frac{2}{5}(x+1)^{5/2}+\frac{2}{3}(x+1)^{3/2}+C\end{aligned}$$