Math 520

U-Substitution

§5.5

This section gives a technique for evaluating indefinite integrals that involve the composition of functions. It is the Chain Rule stated in terms of indefinite integrals.

> A Formula for Integration by Substitution
>
> If f and u are functions such that $f'(u(x)) \cdot u'(x)$ is integrable, then
>
> $$\int f'(u(x)) \cdot u'(x)\, dx = f(u(x)) + C.$$
>
> Your choice of the function u is often inside a composition and whose derivative appears elsewhere.

Example: $\int 2x(x^2+1)^3\, dx$

Solution: Let $\boxed{u = x^2+1}$. Then $\frac{du}{dx} = 2x$ and $\boxed{du = 2x\, dx}$.
Putting the boxed equations together our integral has changed into something we can integrate.

$$\begin{aligned}\int 2x(x^2+1)^3\, dx &= \int u^3\, du \\ &= \frac{u^4}{4} + C \\ &= \frac{(x^2+1)^4}{4} + C\end{aligned}$$

> A Formula for Integration by Substitution
>
> Method of U-Substitution
>
> 1. Pick the u.
> 2. Differentiate u.
> 3. Uncouple the differentials. (Solve for du.)
> 4. Repackage the original problem with u and du.
> 5. Integrate with respect to u.
> 6. Back substitute.

1. $\int 3x^2\sqrt{1+x^3}\, dx$

   Solution: $\frac{2}{3}(x^3+1)^{3/2} + C$

2. $\int 2x\cos(x^2+1)\, dx$

   Solution: $\sin(x^2+1) + C$

3. $\int 2x(x^2+1)^4 \, dx$

Solution: $\frac{1}{5}(x^2+1)^5 + C$

4. $\int 2x \sin(x^2) \, dx$

Solution: $-\cos x^2 + C$

5. $\int (x^3+3x)^4 3(x^2+1) \, dx$

Solution: $\frac{1}{5}(x^3+3x)^5 + C$

6. $\int \frac{3}{3x+1} \, dx$

Solution: $\ln(3x+1) + C$

7. $\int 3x^2\sqrt{1+x^3} \, dx$

Solution: $\frac{2}{3}(x^3+1)^{3/2}$

8. $\int 6xe^{3x^2} \, dx$

Solution: $e^{3x^2} + C$

9. $\int \frac{\cos(\ln x)}{x} \, dx$

Solution: $\sin(\ln x) + C$

10. $\int \sin^3\theta \cos\theta \, d\theta$

Solution: $\frac{\sin^4\theta}{4} + C$