Math 520

Separable Differential Equations

§7.3

We have looked at differential equation from a geometric view point using slope fields. Now we are going to solve certain differential equations algebraically.

> Separable Equation
>
> A **separable equation** is a first-order differential equation in which the expression dy/dx can be factored as a function of x times a function of y. The x's and y's can be separated and written as
>
> $$\frac{dy}{dx} = \frac{g(x)}{f(y)}.$$
>
> To solve this equation we rewrite it in differential form
>
> $$f(y)\,dy = g(x)\,dx$$
>
> so that all the y's are on one side and the x's on the other. Then we can integrate both sides
>
> $$\int f(y)\;dy = \int g(x)\;dx.$$
>
> After integration, if possible, solve for y in terms of x.
>
> The procedure:
>
> 1. Rewrite in differential form. Break up the differential. Put all y's on one side and x's on the other.
> 2. Integrate both sides.
> 3. Consolidate constants to one if necessary.
> 4. Solve for y if possible and consolidate constants to one if necessary.
> 5. Initial value.

1. Solve the differential equation $\dfrac{dy}{dx} = y + xy$ with the initial value of $y(0) = 2$.

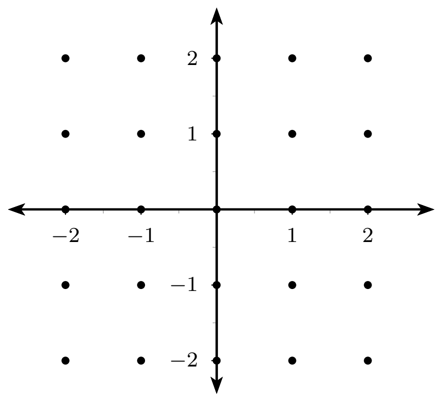
> **Solution:**
>
> $$\begin{aligned}\frac{dy}{dx} &= y + xy\\ \frac{dy}{dx} &= y(1+x)\\ \frac{1}{y}\,dy &= (1+x)\,dx\\ \int \frac{1}{y}\,dy &= \int (1+x)\,dx\\ \ln y &= x + \frac{x^2}{2} + C\end{aligned}$$
>
> We could have used a constant C_1 on the left side and another C_2 on the right side. But then

we could combine these constants by writing $C = C_1 + C_2$. Now, solving for y we have

$$\begin{aligned} \ln y &= x + \frac{x^2}{2} + C \\ e^{\ln y} &= e^{x+\frac{x^2}{2}+C} \\ y &= e^{x+\frac{x^2}{2}+C} \\ y &= e^{x+\frac{x^2}{2}} e^C \\ y &= K e^{x+\frac{x^2}{2}} \end{aligned}$$

2. Consider the differential equation $\dfrac{dy}{dx} = \dfrac{x^2}{y^2}$.

(a) Sketch the slope field of the differential equation.



Solution:

(b) Sketch the solution to the differential equation that satisfies the initial condition $y(0) = 2$.

(c) Solve the differential equation algebraically.

Solution:

$$\begin{aligned} \frac{dy}{dx} &= \frac{x^2}{y^2} \\ y^2 dy &= x^2 dx \\ \int y^2 dy &= \int x^2 dx \\ \frac{1}{3} y^3 &= \frac{1}{3} x^3 + C \end{aligned}$$

Solving for y gives $y = \sqrt[3]{x^3 + 3C} = \sqrt[3]{x^3 + K}$ where $K = 3C$ and since C is an arbitrary constant, so is K.

(d) Find the solution of this equation that satisfies the initial condition $y(0) = 2$.

Solution: Subsituting $x = 0$ into the general solution gives $y = \sqrt[3]{K}$. So, $\sqrt[3]{K} = 2$ and $K = 8$. Therefore the solution is $y = \sqrt[3]{x^3 + 8}$

3. Solve the differential equation $\frac{dy}{dx} = y^2 \cos x$ with the initial condition $y(\pi) = 1$.

Solution:

$$\begin{aligned} \frac{dy}{dx} &= y^2 \cos x \\ \frac{1}{y^2}\, dy &= \cos x\, dx \\ -\frac{1}{y} &= \sin x + C \\ y &= \frac{-1}{\sin x + C} \end{aligned}$$

Now when $y(\pi) = 1$ we have,

$$\begin{aligned} y &= \frac{-1}{\sin x + C} \\ 1 &= \frac{-1}{\sin \pi + C} \\ 1 &= \frac{-1}{C} \\ -1 &= C \end{aligned}$$

So the solution is $y = \frac{-1}{\sin x - 1}$.

4. Solve the differential equation $\frac{dy}{dx} = \frac{6x^2}{2y + \cos y}$.

Solution:

$$\begin{aligned} \frac{dy}{dx} &= \frac{6x^2}{2y + \cos y} \\ (2y + \cos y)\, dy &= 6x^2\, dx \\ \int (2y + \cos y)\, dy &= \int 6x^2\, dx \\ y^2 + \sin y &= 2x^3 + C \end{aligned}$$

It is impossible to solve for y in terms of x so the solution is $y^2 + \sin y = 2x^3 + C$.

5. Solve the differential equation $\frac{dy}{dx} = \frac{y \cos x}{1 + y^2}$ with the initial condition $y(0) = 1$.

Solution:

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{y\cos x}{1+y^2} \\
\frac{1+y^2}{y}\,dy &= \cos x\,dx \\
y^{-1} + y\,dy &= \cos x\,dx \\
\int y^{-1} + y\,dy &= \int \cos x\,dx \\
\ln y + \frac{y^2}{2} &= \sin x + C
\end{aligned}
$$

Now when $y(0) = 1$ we have,

$$
\begin{aligned}
\ln y + \frac{y^2}{2} &= \sin x + C \\
\ln 1 + \frac{1^2}{2} &= \sin 0 + C \\
\frac{1}{2} &= C
\end{aligned}
$$

So the solution is $\ln y + \frac{y^2}{2} = \sin x + \frac{1}{2}$.