Math 520

Related Rates

§4.1

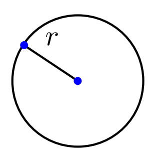
In a related rates problem the ideas is to compute the rate of change of one quantity in terms of the rate of change of another quantity (which may be more easily measured). The procedure is to find an equation that relates the two quantities and then use implicit differentiation and the Chain Rule to differentiate both sides with respect to time.

Strategy for related rates problems:

1. Read and re-read the problem carefully.
2. Draw a diagram of the problem situation when possible. Label parts of the diagram with the appropriate variables and numbers.
3. Introduce notation. Assign symbols to all quantities that are functions of time. Identify all rates of change given and those to be determined.
4. Use a calculus notation to represent the rates, for example, $\frac{dy}{dt}$, $\frac{dA}{dt}$, $\frac{dV}{dt}$, etc.
5. Write an equation that both involves both the variables in step 3 and rates of step 4. (You may need some geometrical formulas to do this.)
6. Differentiate (using implicit differentiation) the equation of step 5.
7. Substitute all known values into the differentiated equation and solve for the desired unknown rate or quantity.

1. A spherical snowball melts in such a way that the instant at which its radius is 20 cm, its radius is decreasing at 3 cm/min. Determine at what rate the volume of the ball of snow changing at that instant?

Solution:



The equation of the volume of a sphere is: $V = \frac{4}{3}\pi r^3$

Because the snowball is melting, **both the radius and volume are really functions of time**. Differentiate the volume formula with respect to time, t.

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

Substituting all known quantities into this equation and solve for the desired rate.

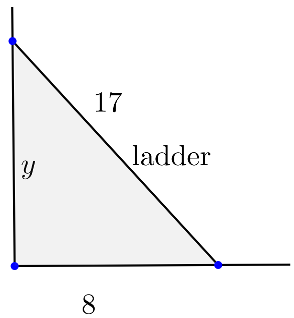
We are given the when $r = 20$, $\frac{dr}{dt} = -3$, thus,

$$\frac{dV}{dt} = 4\pi(20)^2(-3)$$
$$\frac{dV}{dt} = -4800\pi$$

2. A 17 foot ladder is leaning against a wall. The base of the ladder is moving away from the all at a constant rate of 2 feet per second.

   (a) What is the rate at which the top of the ladder is moving when the base of the ladder is 8 feet from the wall?

Solution: First we'll draw a picture and introduce notation.

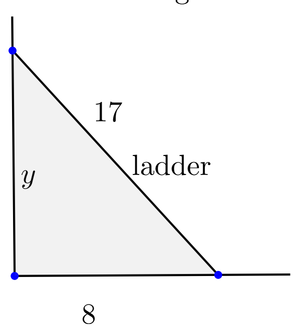


$\frac{dx}{dt} = 2$ feet per second and using the Pythagorean Theorem, when $x = 8$, $y = 15$.

The equation that relates all these is:
$x^2 + y^2 = 17^2$
Differentiating implicitly since both x and y are functions of t we have: $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$. Substituting the given information into the resulting equation gives us $8(2) + 15\frac{dy}{dt} = 0$.
So, $\frac{dy}{dt} = \frac{-16}{15}$ ft/sec.

   (b) Consider the triangle formed by the side of the building, the ladder, and the ground. What is the rate at which the area of the triangle is changing when the base of the ladder is 8 feet from the wall?

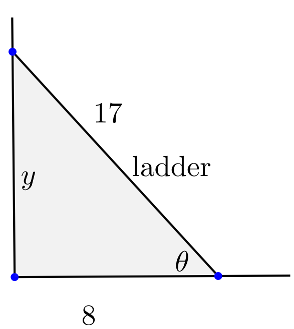
Solution: Again we'll draw a picture and introduce notation. $A = \frac{1}{2}xy$. So,



$$\begin{aligned}
\frac{dA}{dt} &= \frac{1}{2}\left[x\frac{dy}{dt} + y\frac{dx}{dt}\right] \\
&= \frac{1}{2}\left[8\left(\frac{-16}{15}\right) + 15(2)\right] \\
&= \frac{161}{15} \text{ feet/sec}
\end{aligned}$$

   (c) Consider the angle created between the ladder and the ground. What is the rate at which this angle is changing when the base of the ladder is 8 feet from the wall.

Solution: Draw a picture and introduce notation. $\sin\theta = \frac{y}{17}$. So, $17\sin\theta = y$.

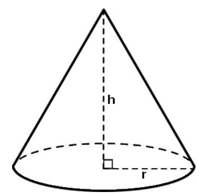


Differentiating both sides gives,

$$\begin{aligned}
17\cos\theta\frac{d\theta}{dt} &= \frac{dy}{dt} \\
\frac{d\theta}{dt} &= \frac{\frac{-16}{15}}{17\left(\frac{8}{17}\right)} \\
&= \frac{\frac{-16}{15}}{8} \\
&= \frac{-2}{15} \text{ radians/sec}
\end{aligned}$$

3. If dry sand spills to form a conical pile with height equal to the radius of the base, how fast is the height increasing when the height is 3 ft and sand is being added at the rate of 2 cu ft/min?

Solution:



The volume of a cone is

$$V = \frac{1}{3}\pi r^2 h.$$

Since $h = r$ the new formula for the Volume of this cone is

$$V = \frac{1}{3}\pi h^3.$$

Differentiating both sides with respect to t gives us,

$$\frac{dV}{dt} = \pi h^2 \frac{dh}{dt}.$$

At the moment of concern $\frac{dV}{dt} = 2$ and $h = 3$. Substituting this into the derivative we get...

$$\begin{aligned} \frac{dV}{dt} &= \pi h^2 \frac{dh}{dt} \\ 2 &= \pi (3)^2 \frac{dh}{dt} \\ \frac{2}{9\pi}\text{ft/min} &= \frac{dh}{dt} \end{aligned}$$