Math 520

Optimization

§4.6

An optimization problem seeks to find the largest (or smallest) value of a quantity (such as maximum revenue or minimum surface area) given certain limits or constraints. The problems arise in many areas of application. There is a particular method to follow that will help set up and solve them.

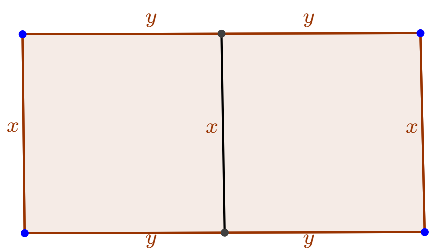
A procedure to solve optimization problems

1. Draw a diagram.
2. Introduce notation and identify what quantity is being maximized (or minimized).
3. Write an equation that expresses the quantity being maximized (or minimized) as a function of just one variable.
4. Find the absolute extreme of the function in (3)

1. A gardener wishes to create two equal sized gardens by enclosing a rectangular area with 300 feet of fencing and fence it down the middle. What is the largest rectangular area that may be enclosed?

Solution:

Step 1: Let A be the area of the enclosed



Label one side x and the half side of the other as y since that side is bisected.

Step 2: $A = x(2y) = 2xy$. The total amount of fence is $3x + 4y = 300$. Thus, $y = \dfrac{300-3x}{4}$ and $x \in [0, 100]$.

Step 3: Therefore,

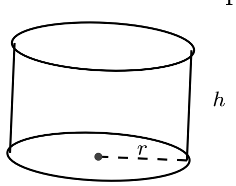
$$A = 2xy = 2x\frac{300-3x}{4} = 150x - \frac{3}{2}x^2.$$

Step 4: Maximize $A = 150x - \frac{3}{2}x^2$ on $[0, 100]$. $A' = 150 - 3x = 0$ when $x = 50$.

Then $y = \dfrac{300-3(50)}{4} = 37.5$.

So the maximum value of A is $2xy = 2(50)(37.5) = 3750$.

2. A cylindrical cup is to be made from 12 square inches of aluminum. What is the largest possible volume of this cup?



Solution:

Step 1: Maximuze the volume V of a cup whose circular base has radius r and whose height is h Step 2: $V = \pi r^2 h$. The total surface area (bottom plus side) is $12 = \pi r^2 + 2\pi r h$. So,

$$h = \frac{12 - \pi r^2}{2\pi r} \quad \text{where} \quad r \in \left[0, \sqrt{\frac{12}{\pi}}\right]$$

Step 3: So

$$V = \pi r^2 \frac{12 - \pi r^2}{2\pi r} = 6r - \frac{\pi}{2} r^3$$

Step 4: $V' = 6 - \frac{3\pi}{2} r^2$. So,

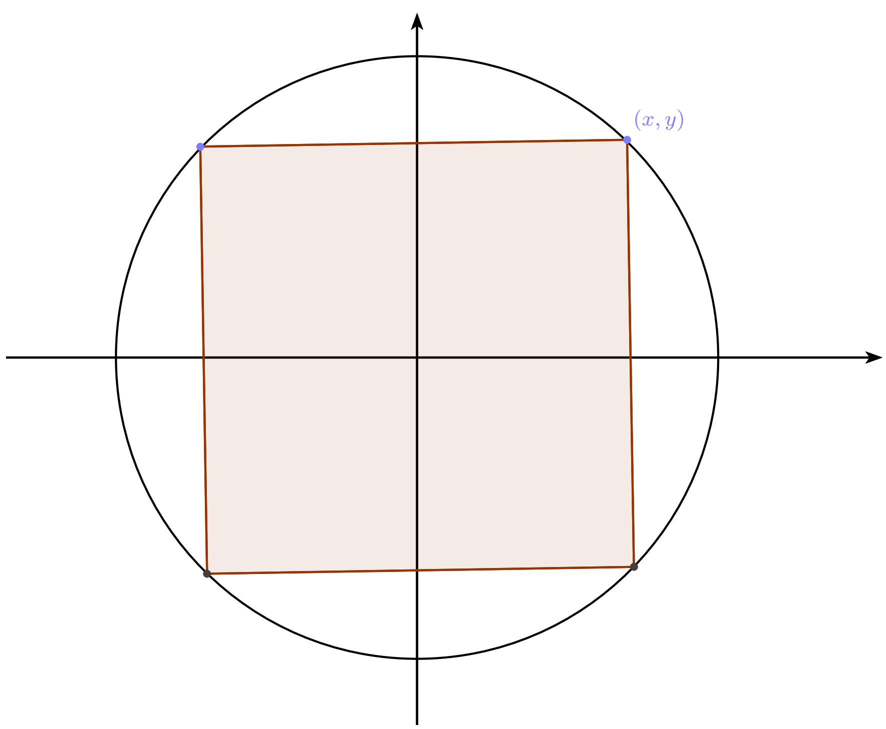
$$\begin{aligned} 6 - \frac{3\pi}{2} r^2 &= 0 \\ r^2 &= \frac{4}{\pi} \\ r &= \frac{2}{\sqrt{\pi}} \end{aligned}$$

$V'' = -3\pi r$ so $r = \frac{2}{\sqrt{\pi}}$ is a maximum. At $r = \frac{2}{\sqrt{\pi}}$ the maximum volume is

$$V = 6\left(\frac{2}{\sqrt{\pi}}\right) - \frac{\pi}{2}\left(\frac{2}{\sqrt{2}}\right)^3 = \frac{8}{\sqrt{\pi}}.$$

3. Find the dimensions of the largest rectangular peg that can be put into a round hole with diameter 10 cm.

Solution: Step 1: The largest peg means one with the largest cross sectional area. The cross section of the peg in the hole is shown in the figure below.



Step 2: The quantity to be maximized is area

$$A = (2x)(2y) = 4xy.$$

The equation for the hole is

$$x^2 + y^2 = 25$$

Therefore

$$y = \sqrt{25 - x^2} \quad \text{where} \quad x \in [0, 5]$$

Step 3:

$$A(x) = 4xy = 4x\sqrt{25 - x^2}$$

Step 4: Maximize $A(x) = 4xy = 4x\sqrt{25 - x^2}$ on $[0, 5]$. Both 0 and 5 produce a minimum area of zero, so the maximum occurs when $A'(x) = 0$. Graphically we see that $A'(x) = 0$ when $x = \frac{5\sqrt{2}}{2}$. Yes, this makes a square peg.