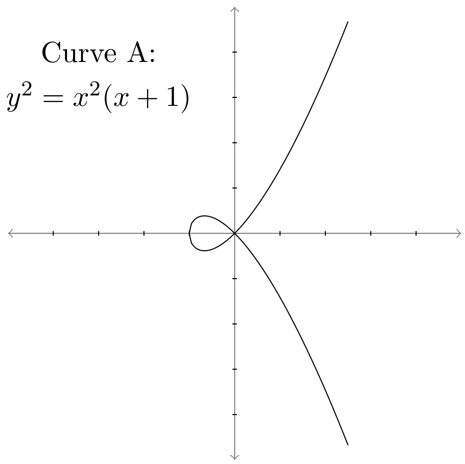
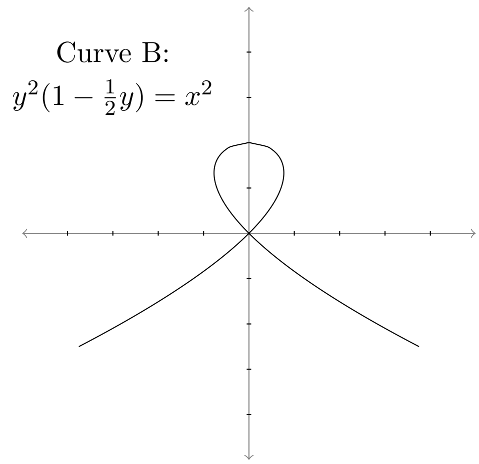
Math 520

More Implicit Differentiation and Inverse Trig Differentiation

§3.6





1. (a) Use implicit differentiation to find all the points in Curve A with a horizontal tangent line. (Looking at the graph, how many such points should there be?)

> **Solution:** Using implicit differentiation, we get $\frac{dy}{dx} = \frac{x(3x+2)}{2y}$, so the points where $dy/dx = 0$ are those with x-coordinate $-\frac{2}{3}$. (We can't set $x = 0$, because then the equation for Curve A says that y is also 0. Then dy/dx has a zero in the denominator.) Solving for y, we get the two points $\left(-\frac{2}{3}, \frac{4}{27}\right)$ and $\left(-\frac{2}{3}, -\frac{4}{27}\right)$. ($\frac{4}{27} \approx .148148....$)

(b) Use implicit differentiation to find all the points in Curve B with a horizontal tangent line. (Looking at the graph, how many such points should there be?)

> **Solution:** Using implicit differentiation, we get
>
> $$\frac{dy}{dx} = \frac{2x}{2y - \frac{3}{2}y^2},$$
>
> so the points where $dy/dx = 0$ are those with $x = 0$ and y nonzero. If $x = 0$ on the curve, then either $y = 0$ or $y = 2$, so we have only the point $(0, 2)$.

(c) Try to find $\frac{dy}{dx}$ at the point $(0, 0)$ on both graphs. What goes wrong?

> **Solution:** It's not possible to plug in $x = 0$ and $y = 0$ to either of the expressions for $\frac{dy}{dx}$. The derivative function tells us nothing about $(0, 0)$.

2. This next question is a new type of problem that you can solve now that you know about implicit differentiation. Suppose a snowball is rolling down a hill, and its radius r is growing at a rate of 1 inch per minute. The volume V of the snowball grows more quickly as the snowball gets bigger. In this question, you'll find the rate of change of the volume, $\frac{dV}{dt}$, at the instant when the radius r is 6 inches.

   (a) First, apply geometry to the situation. Can you think of an equation that relates the variables r and V to each other?

   Solution: $V = \frac{4}{3}\pi r^3$.

   (b) Now the variables V and r change as time changes, so we can think of them as functions of t. Differentiate the equation you came up with in part (a) with respect to t.

   Solution: $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$.

   (c) What is the rate of change of the radius? Use this to simplify your equation from part (b).

   Solution: The rate is given; $\frac{dr}{dt} = 1$. (Units: inches/minute.)

   (d) What is the rate of change of V when the radius of the snowball is 6 inches?

   Solution: $\frac{dV}{dt} = 4\pi(6)^2 = 144\pi$. (Units: cubic inches/minute.)

Here are the formulas for the derivatives of the inverse trigonometric functions:

$$\frac{d}{dx}\sin^{-1} = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\csc^{-1} x = \frac{-1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}\cos^{-1} = \frac{-1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\sec^{-1} x = \frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}\tan^{-1} = \frac{1}{1+x^2} \qquad \frac{d}{dx}\cot^{-1} x = \frac{-1}{1+x^2}$$

3. Find the derivatives of the following

   (a) $y = \sin^{-1}(x^3)$

   Solution: $y' = \dfrac{3x^2}{1-x^6}$

   (b) $y = (\tan^{-1} x)^3$

   Solution: $y' = 3(\tan^{-1} x)^2 \cdot \frac{1}{1+x^2}$

   (c) $y = x\cos^{-1} x$

   Solution: $y' = \dfrac{-x}{\sqrt{1-x^2}} + \cos^{-1} x$

   (d) $y = e^{\csc^{-1}(x^2)}$

   Solution: $y' = e^{\csc^{-1}(x^2)}\left(\dfrac{-1}{x^2\sqrt{(x^2)^2-1}}\right)\cdot 2x = \dfrac{-2e^{\csc^{-1}(x^2)}}{x\sqrt{x^4-1}}$