Math 520

Max & Min Values

§4.2

This section gives you some tools for finding the largest and smallest values of $f(x)$ for a given function f. The largest and smallest values of $f(x)$ are called **absolute extreme**. Those points where $f(x)$ is largest or smallest in a region around the point x are called relative extreme.

Absolute Extrema:
A function f has an **absolute maximum** at c when

$$f(x) \leq f(c) \quad \text{for all } c.$$

Thus, a highest point on the graph occurs at $(c, f(c))$.

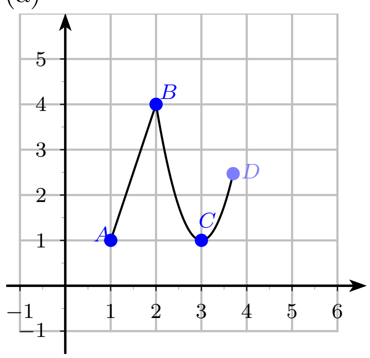
A function f has an **absolute minimum**at c when

$$f(c) \leq f(x) \quad \text{for all } c.$$

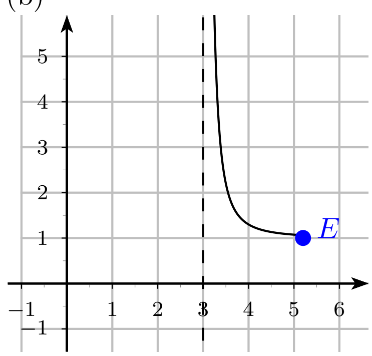
Thus, a lowest point on the graph occurs at $(c, f(c))$.

1. In each of the graphs below, identify the *extreme values* (absolute max and min) of f.

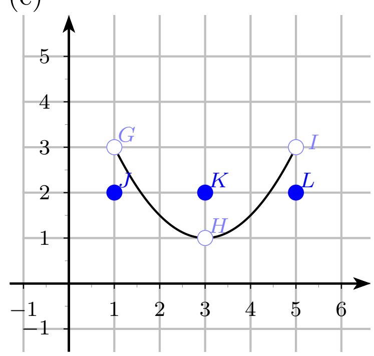
(a)



(b)



(c)



Solution: (a) absolute min at A and C; absolute max at B
(b) absolute min at E, no absolute max
(c) no absolute max or min.

Relative Extrema:
If there is some open interval I containing c such that

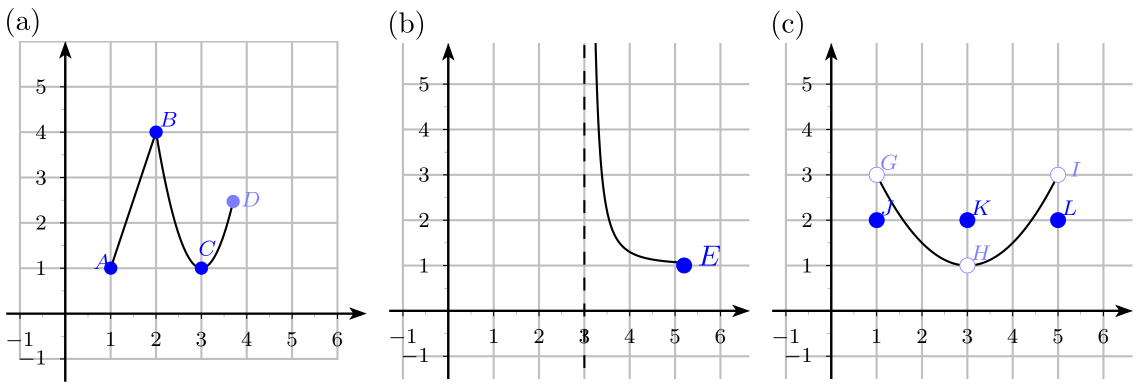
$$f(x) \leq f(c) \qquad \text{for all } c,$$

then f has a **local maximum** at c. If there is some open interval I containing c such that

$$f(c) \leq f(x) \qquad \text{for all } c,$$

then f has a **local minimum** at c.

2. In each of the graphs below, identify the *relative extreme* (local max and min) of f.



Solution: (a) local min at C; local max at B
(b) no local max or min
(c) local max at K

Critical Numbers:
A **critical number** c is a number in the domain of f for which either $f'(c) = 0$ or $f'(c)$ does not exist.

Determining where $f'(c) = 0$ is a matter of solving the equation $f'(x) = 0$.

3. Find all the critical points of the following:
   (a) $f(x) = 3x^4 + 20x^3 - 36x^3 + 7$

   Solution: $f'(x) = 12x^3 + 60x^2 - 72x = 12x(x-1)(x+6)$
   So, $f'(x) = 0$ when $x = 0,\ 1,\ -6$

   (b) $f(x) = 9\sqrt{x} + x^{3/2}$

   Solution: $f'(x) = \dfrac{9}{2\sqrt{x}} + \dfrac{3\sqrt{x}}{2}$ and $f'(x)$ does not exist at $x = 0$.

How to Find Extrema:
The extreme values of a continuous function f on a closed interval $[a, b]$ always exist.
The occur either at a, at b, or at a critical number of f in (a, b).

To find absolute maximum and minimum values of f on a closed interval $[a, b]$, use the **Closed Interval Method**.

1. Find $f(x)$ at the critical numbers x in (a, b).
2. Find $f(a)$ and $f(b)$.
3. Choose the largest and smallest values from the results of steps 1 and 2.

4. Find the extreme values of $f(x) = \sqrt{10x - x^2}$ on $[2, 10]$.

Solution: Since f is continuous on $[2, 10]$, the extreme values occur at 2, 10, or some critical number between 2 and 10.

$$f'(x) = \frac{1}{2}(10x - x^2)^{-1/2}(10 - 2x) = \frac{5 - x}{\sqrt{10x - x^2}}$$

On $[2, 10]$,
$f'(x) = 0$ at $x = 5$
$f'(x)$ does not exist at $x = 10$.
$f(2) = 4 \qquad f(5) = 5 \qquad f(10) = 0$
So, the absolute minimum is 0 and occurs at $x = 10$
The absolute maximum is 5 and occurs at $x = 5$.

5. Find the extreme values of $f(x) = 2x + \sin x$ on $[0, 2\pi]$.

Solution: $f'(x) = 2 + \cos x$. Solving $f'(x) = 0$ yields $\cos x = -2$ which has no solutions. Therefore there are not critical points. So,
$f(0) = 2(0) + \sin 0 = 0$
$f(2\pi) = 2(2\pi) + \sin 2\pi = 4\pi$
So, the absolute minimum is 0 and the absolute maximum is 4π.