Math 520

Derivatives of Log Functions & Log Differentiation

§3.7

This section shows you how to find the derivative of logarithmic functions and provides a techniques (logarithmic Differeintiation) for finding the derivative of certain complicated functions.

Recall that $y = \log_a x$ is logically equivalent to $a^y = x$. We can differentiate $a^y = x$ implicitly...

Step 1: $(a^y \cdot \ln a)y' = 1$ because $\frac{d}{dx}a^x = a^x \ln a$

Step 2: $y' = \dfrac{1}{a^y \cdot \ln a} = \dfrac{1}{x \ln a}$ because $a^y = x$.

Recall $\log_e x = \ln x$. To get the formula for the derivative of $\ln x$ we substitute $x = e$ to get...

$$\frac{d}{dx} \ln x = \frac{d}{dx} \log_e x = \frac{1}{x \ln e} = \frac{1}{x}$$

$$\frac{d}{dx} \log_a x = \frac{1}{x \ln a} \qquad \frac{d}{dx} \log_a g(x) = \frac{1}{g(x) \ln a} \cdot g'(x)$$

$$\frac{d}{dx} \ln x = \frac{1}{x} \qquad \frac{d}{dx} \ln g(x) = \frac{1}{g(x)} \cdot g'(x)$$

1. Determine y':

   (a) $y = \ln x^2$

   Solution: $y' = \dfrac{1}{x^2} \cdot 2x = \dfrac{2}{x}$

   (b) $y = \ln \cos x$

   Solution: $y' = \dfrac{1}{\cos x} \cdot - \sin x = - \tan x$

   (c) $y = x^2 \ln x$

   Solution: $y' = x^2 \left(\dfrac{1}{x} \right) + \ln x(2x) = x + 2x \ln x$

   (d) $y = \log_2(x^2 + 1)$

   Solution: $y' = \dfrac{1}{(x^2 + 1) \ln 2} \cdot 2x$

(e) $y = \ln\sqrt{\dfrac{x^2+1}{2x^3}}$

Solution: Let's begin by simplifying...

$$\begin{aligned} y = \ln\sqrt{\frac{x^2+1}{2x^3}} &= \ln\left(\frac{x^2+1}{2x^3}\right)^{1/2} \\ &= \frac{1}{2}\ln\left(\frac{x^2+1}{2x^3}\right) \\ &= \frac{1}{2}(\ln(x^2+1) - \ln(2x^3)) \\ &= \frac{1}{2}\ln(x^2+1) - 3\ln 2x. \end{aligned}$$

Then,

$y' = \dfrac{1}{2}\left(\dfrac{2x}{x^2+1} - \dfrac{3}{x}\right) = \dfrac{x}{x^2+1} - \dfrac{3}{2x}$

The derivative of a function $y = f(x)$ can be found using **logarithmic differentiation**. Here are the steps...

Step 1: Take the ln of both sides of $y = f(x)$ to get $\ln y = \ln(f(x))$.

Step 2: Simplify $\ln(f(x))$ using the properties of logarithms.

Step 3: Differentiate both sides implicitly and solve for y'.

Properties of Logarithms

$$\ln e = 1 \qquad \ln 1 = 0$$

$$\ln(ab) = \ln a + \ln b \qquad \ln\left(\frac{a}{b}\right) = \ln a - \ln b$$

$$\ln b^a = a\ln b$$

2. Use logarithmic differentiation to find y'.

(a) $y = x^{\frac{2}{x}}$

Solution:

Step 1: $\ln y = \ln x^{\frac{x}{2}}$

Step 2: $\ln y = \frac{2}{x}\ln x$

Step 3:

$$\begin{aligned} \frac{1}{y}y' &= \frac{2}{x}\left(\frac{1}{x}\right) + \ln x\left(-\frac{2}{x^2}\right) \\ \frac{1}{y}y' &= \frac{2}{x^2}(1 - \ln x) \\ y' &= y\left(\frac{2}{x^2}(1-\ln x)\right) \\ y' &= x^{\frac{2}{x}}\left(\frac{2}{x^2}(1-\ln x)\right) \\ y' &= 2x^{\frac{2}{x}-2}(1-\ln x) \end{aligned}$$

(b) $y = \left(\dfrac{10x^3}{\sqrt{x+1}}\right)^4$

Solution:

Step 1: $\ln y = \ln\left(\frac{10x^3}{\sqrt{x+1}}\right)^4$

Step 2:

$$\begin{aligned}\ln y &= \ln\left(\frac{10x^3}{\sqrt{x+1}}\right)^4 \\ &= 4(\ln 10x^3 - \ln\sqrt{x+1})\end{aligned}$$

Step 3:

$$\frac{y'}{y} = 4\left(\frac{30x^2}{10x^3} - \frac{\frac{1}{2}(x+1)^{-\frac{1}{2}}}{(x+1)^{\frac{1}{2}}}\right)$$

$$\frac{y'}{y} = 4\left(\frac{3}{x} - \frac{1}{2}(x+1)^{(-\frac{1}{2}-\frac{1}{2})}\right)$$

$$\frac{y'}{y} = 4\left(\frac{3}{x} - \frac{1}{2(x+1)}\right)$$

$$y' = y \cdot 4\left(\frac{3}{x} - \frac{1}{2(x+1)}\right)$$

$$y' = \left(\frac{10x^3}{\sqrt{x+1}}\right)^4 \cdot 4\left(\frac{3}{x} - \frac{1}{2(x+1)}\right)$$