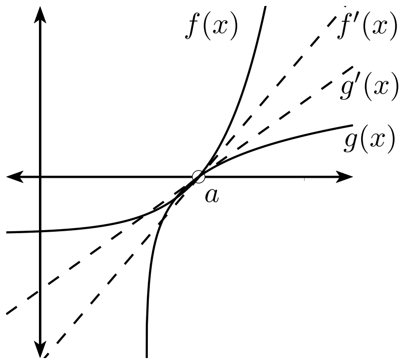
Math 520

Indeterminant Forms and L'Hospital's Rule

§4.5

Sometimes applying the limit theorems of chapter 3 to a limit results in nonsensical expressions such as $\frac{0}{0}$ even though the limit exists. For example, applying the limit theorems to $\lim_{x\to 2}\frac{x^2-4}{x-2}$ yields $\frac{0}{0}$; we know the limit exists and is 4 (you need to factor first to find this limit). L'Hopspital's Rule is a method for evaluating limits that have indeterminate forms such as $\frac{0}{0}$.



L'Hospital's Rule says that if $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$, the $\lim_{x\to a}\frac{f(x)}{g(x)}$ may be determined by finding the limit of the quotient of the derivatives. In essence,

$$\frac{f(x)}{g(x)} \approx \frac{f'(x)}{g'(x)}$$

for x near a.

There are seven such forms discussed in this section.

Indeterminate forms $\frac{0}{0}$ and $\frac{\infty}{\infty}$

L'Hospital's Rule for $\frac{0}{0}$

- If $\lim\limits_{x\to a} f(x) = 0$ and $\lim\limits_{x\to a} g(x) = 0$, then $\lim\limits_{x\to a}\frac{f(x)}{g(x)} = \lim\limits_{x\to a}\frac{f'(x)}{g'(x)}$

L'Hospital's Rule for $\frac{\infty}{\infty}$

- If $\lim\limits_{x\to a} f(x) = \infty$ and $\lim\limits_{x\to a} g(x) = \infty$, then $\lim\limits_{x\to a}\frac{f(x)}{g(x)} = \lim\limits_{x\to a}\frac{f'(x)}{g'(x)}$

Example:

$$\lim_{x\to 2}\frac{x^2-4}{x-2} \overset{\text{H}}{=} \lim_{x\to 2}\frac{2x}{1} = 4$$

Indeterminate form $0\cdot\infty$

To evaluate $\lim\limits_{x\to a} f(x)\cdot g(x)$ which has indeterminate form $0\cdot\infty$, first rewrite $f(x)\cdot g(x)$ as either

$$\frac{f(x)}{\frac{1}{g(x)}} \quad \text{or} \quad \frac{g(x)}{\frac{1}{f(x)}}$$

Now apply L'Hospital's Rule.

Example:

$$\lim_{x\to\infty} xe^{-x} = \lim_{x\to\infty}\frac{x}{e^x} \overset{\text{H}}{=} \lim_{x\to\infty}\frac{1}{e^x} = 0$$

Indeterminate form $\infty - \infty$

To evaluate $\lim_{x\to a} f(x) - g(x) = \infty - \infty$ rewrite $f(x) - g(x)$ as a single term (ie. get a common denominator or use all sines and cosines). Now apply L'Hospital's Rule.

Example:

$$\begin{aligned}\lim_{x\to\frac{\pi}{2}^-} (\sec x - \tan x) &= \lim_{x\to\frac{\pi}{2}^-} \left(\frac{1}{\cos x} - \frac{\sin x}{\cos x}\right) \\ &= \lim_{x\to\frac{\pi}{2}^-} \frac{1-\sin x}{\cos x} \\ &\overset{\text{H}}{=} \lim_{x\to\frac{\pi}{2}^-} \frac{-\cos x}{-\sin x} \\ &= 0\end{aligned}$$

Indeterminate form 0^0, 1^∞, and ∞^0

To evaluate $\lim_{x\to a} (f(x))^{g(x)}$ where indeterminate forms of type 0^0, 1^∞, and ∞^0 are possible,

1. Let $y = f(x)^{g(x)}$
2. Then $\ln y = g(x) \cdot \ln f(x)$
3. If $\lim_{x\to a} g(x) \cdot \ln f(x)$ (type $0 \cdot \infty$) exists and equals L, then

$$\lim_{x\to a} (f(x))^{g(x)} = e^L$$

Example: $\lim_{x\to 0^+} x^x$ is type 0^0. So let $y = x^x$. Then $\ln y = \ln x^x = x \ln x$. Now,

$$\begin{aligned}\lim_{x\to 0^+} x\ln x &= \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} (\text{type } \frac{\infty}{\infty}) \\ &\overset{\text{H}}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{\frac{-1}{x^2}} \\ &= \lim_{x\to 0^+} -x \\ &= 0\end{aligned}$$

Thus, $\lim_{x\to 0^+} x^x = e^0 = 1$.

1. $\lim_{x\to 3} \dfrac{x^2 - 7x + 12}{x^2 - 9}$

Solution: type $\frac{0}{0}$

$$\begin{aligned}\lim_{x\to 3} \frac{x^2 - 7x + 12}{x^2 - 9} &\overset{\text{H}}{=} \lim_{x\to 3} \frac{2x - 7}{2x} \\ &= -\frac{1}{6}\end{aligned}$$

2. $\lim_{x\to 0} \dfrac{\sin x}{x}$

Solution: type $\frac{0}{0}$

$$\lim_{x\to 0}\frac{\sin x}{x} \overset{\text{H}}{=} \lim_{x\to 0}\frac{\cos x}{1}$$
$$= \frac{1}{1} = 1$$

3. $\displaystyle\lim_{x\to\infty}\frac{e^x}{3x}$

Solution: type $\frac{\infty}{\infty}$

$$\lim_{x\to\infty}\frac{e^x}{3x} \overset{\text{H}}{=} \lim_{x\to\infty}\frac{e^x}{3}$$
$$= \infty$$

4. $\displaystyle\lim_{x\to 0^+} x\cot x$

Solution: type $0\cdot\infty$

So, we need to rewrite $x\cot x = x\cdot\frac{\cos x}{\sin x} = \frac{x}{\sin x}\cdot\cos x$. Now, since $\displaystyle\lim_{x\to 0^+}\frac{x}{\sin x}$ has form $\frac{0}{0}$ we can evaluate the limit as follows.

$$\begin{aligned}\lim_{x\to 0^+} x\cot x &= \lim_{x\to 0^+}\left(\frac{x}{\sin x}\cdot\cos x\right)\\ &= \lim_{x\to 0^+}\frac{x}{\sin x}\cdot\lim_{x\to 0^+}\cos x\\ &\overset{\text{H}}{=} \lim_{x\to 0^+}\frac{1}{\cos x}\cdot\lim_{x\to 0^+}\cos x\\ &= 1\cdot 1 = 1\end{aligned}$$