Math 520

Implicit Differentiation

§3.6

This section shows you a way to find the derivative of a function even when you are not given the rule for the function explicitly. The rule will be determine by an equation. This section also shows you how to find the derivatives of the inverse trigonometric functions.

Given an equation that defines y as in implicit function of x, $\frac{dy}{dx}$ may be found by the technique of **Implicit Differentiation**:

Step 1: Take the derivative with respect to x of both sides of the equation.
Step 2: Solve the result for y' or $\frac{dy}{dx}$.

Important: Be sure to remember that y is a function of x, so that when a term such as y^3 is differentiated, the result is $3y^2y'$ (by the Chain Rule) and not just $3y^2$.

1. Find $\frac{dy}{dx}$ implicitly:

   (a) $x^2y^3 = 2x + 1$

   Solution:
   Step 1: Differentiate both sides with respect to x using the Product Rule because both x^2 and y^3 are function of x to get

   $$x^2(3y^2y') + y^3(2x) = 2.$$

   Step 2: Solve for y':

   $$\begin{aligned} x^2(3y^2y') + y^3(2x) &= 2 \\ x^2(3y^2y') &= 2 - y^3(2x) \\ y' &= \frac{2 - 2xy^3}{3x^2y^2} \end{aligned}$$

   (b) $3x^2 - 5xy + y^2 = 10$

   Solution:
   Step 1: Differentiate both sides to get

   $$6x - (5x(y') + 5y) + 2yy' = 0.$$

   Step 2: Solve for y':

   $$\begin{aligned} 6x - (5x(y') + 5y) + 2yy' &= 0 \\ 5xy' + 2yy' &= -5y - 6x \\ (2y - 5x)y' &= 5y - 6x \\ y' &= \frac{5y - 6x}{2y - 5x} \end{aligned}$$

2. Find the slope of the tangent line to the curve defined by

$$x^2 + 2xy - y^2 = 1$$

at the point $(5, 2)$.

Solution: First find $\dfrac{dy}{dx}$ implicitly:
Step 1: Differentiate both sides to get

$$2x + (2xy' + 2y) - 2yy' = 0.$$

Step 2: Solve for y':

$$\begin{aligned} 2x + (2xy' + 2y) - 2yy' &= 0 \\ 2y'(x - y) &= -2(x + y) \\ y' &= -\frac{x + y}{x - y} \end{aligned}$$

Now use $x = 5$ and $y = 2$ so

$$\frac{dy}{dx} = -\frac{5 + 2}{5 - 2} = -\frac{7}{3}.$$

3. At what points does the implicitly defined curve

$$y^2 = x^3 + 3x^2$$

have a horizontal tangent?

Solution: It will have a horizontal tangent when the derivative is 0. That is we need to solve $\frac{dy}{dx} = 0$.
Step 1: Differentiate both sides to get $2yy' = 3x^2 + 6x$
Step 2: Solve for y'

$$y' = \frac{3x^2 + 6x}{2y}$$

Now, $\frac{dy}{dx} = 0$ when the numerator is 0. Thus,

$$\begin{aligned} 3x^2 + 6x &= 0 \\ 3x(x + 2) &= 0 \end{aligned}$$

So, when $x = 0$ and $x = -2$. Now we need to find the y values by substituting back into the original equation to get...
for $x = 0$ $\quad y^2 = 0^3 + 3(0)^2 = 0$
for $x = -2$ $\quad y^2 = (-2)^3 + 3(-2)^2 = -8 + 12 = 4$ So, $y = \pm 2$.

But at the point $(0, 0)$, the derivative does not exist! (Plug this point into y' and you get $\frac{0}{0}$.)
Thus the curve has a horizontal tangent at $(-2, 2)$ and $(-2, -2)$.

The derivatives of inverse trigonometric functions can be found by the technique of Implicit Differentiation.

$$\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}$$

and

$$\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2}$$

4. Find $\dfrac{d}{dx}\cos^{-1}(x)$.

Solution: Recall the definition of the arccos function:

$$y = \cos^{-1} x \qquad \text{means} \qquad \cos y = x$$

So, we can differentiate $\cos y = x$ implicitly with respect to x and get

$$(\sin y)y' = 1 \qquad \text{or} \qquad y' = \frac{1}{\sin y}$$

Using the Pythagorean Trig Identity, $\sin^2 y + \cos^2 y = 1$, we have $\sin y = \sqrt{1-\cos^2 y}$. So,

$$y' = \frac{1}{\sin y} = \frac{1}{1-\sqrt{\cos^2 y}} = \frac{1}{\sqrt{1-x^2}}$$

5. Find the derivative of $y = \sin^{-1} x^2$

Solution: Using the formula from above and the Chain Rule,

$$y' = \frac{1}{1-(x^2)^2} \cdot (2x) = \frac{2x}{1-x^4}$$