Math 520

Review of Fundamental Theorem of Calculus

§5.3

The Fundamental Theorem of Calculus relates derivatives and integrals.

The Fundamental Theorem of Calculus

If f is a continuous function on $[a, b]$ and

$$g(x) = \int_a^x f(t)\, dt \text{ , then } g'(x) = f(x).$$

If f is a continuous function on $[a, b]$ and F is any antiderivative of f, then

$$\int_a^b f(x)\, dx = F(b) - F(a)$$

1. Find a polynomial function f with all the following properties:
   - $f'(x) = ax^2 + bx$
   - $f'(1) = 6$
   - $f''(1) = 18$
   - $\int_1^2 f(x)\, dx = 18$

Solution: $f'(1) = a(1)^2 + b(1) = a + b = 6$.
$f''(1) = 2a(1) + b(1) = 2a + b = 18$. Solving these two equations for both a and b gives $a = 12$ and $b = -6$.
Now,

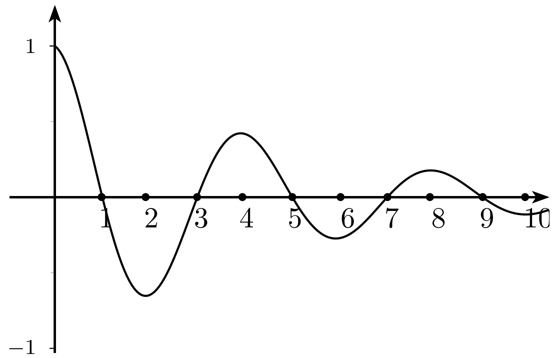
$$\begin{aligned} f(x) &= \int f'(x)\, dx \\ &= \int ax^2 + bx\, dx \\ &= \frac{ax^3}{3} + \frac{bx^2}{2} + c \\ &= \frac{12x^3}{3} + \frac{-6x^2}{2} + c \\ &= 4x^3 + 3x^2 + c \end{aligned}$$

Since $\int_1^2 f(x)\, dx = 18$ we have,

$$\begin{aligned} \int_1^2 4x^3 + 3x^2 + C\, dx &= x^4 - x^3 + Cx \Big|_1^2 \\ &= (2^4 - 2^3 + 2C) - (1 - 1 + C) \\ &= 16 - 8 + 2C - C \\ &= 8 + C \end{aligned}$$

And $8 + c = 18$. So, $C = 10$. Thus, $f(x) = 4x^3 - 3x^3 + 10$.

2. Let $g(x) = \int_0^x f(t)\, dt$ where $f(t)$ is the function shown below



(a) Does $g(x)$ have any local maxima in the interval $[0, 10]$? If so, where are they located?

Solution: yes, at $x = 1$, $x = 5$, and $x = 9$

(b) Does $g(x)$ have any local minima in the interval $[0, 10]$? If so, where are they located?

Solution: yes, at $x = 3$ and $x = 7$

(c) At what value of x does $g(x)$ attain its minimum value on the interval $[1, 10]$?

Solution: $x = 3$

(d) On which subinterval(s) of $[0, 10]$, if any, is the graph of $g(x)$ concave up? Justify your answer.

Solution: g is concave up on $(2, 4)$ and $(6, 8)$ because that is when g'' is positive.