Math 520

Average Value of a Function

§6.4

As the graph of a function f varies from $x = a$ to $x = b$, $f(x)$ may assume many different values. This section shows how the average of these values may be determined. This section also contains a restatement of the Mean Value Theorem—this time in terms of definite integrals.

The Average Value of a Function

The average value of a continuous function $y = f(x)$ over a closed interval $[a, b]$ is

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

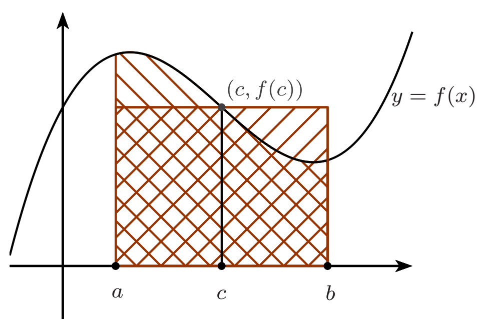
On way to think of this is that $\int_a^b f(x)\,dx$ is the are under $f(x)$ and so dividing it by the width $(b-a)$ of the area yields the (average) height of $f(x)$.

The Mean Value Theorem for Integrals

If f is a continuous on a closed interval $[a, b]$ there exists $x \in [a, b]$ such that

$$\int_a^b f(x)\,dx = f(c)(b-a).$$

For $f(x) \geq 0$, we can think of $f(c)(b-a)$ as the area of a rectangle with height $f(c)$ and width $b-a$. The Mean Value Theorem for Integrals says there is some point $c \in [a, b]$ so that the are of this rectangle equals the area under the curve.



1. Let $f(x) = (x-3)^2$.

   (a) Find the average value of f on $[2, 5]$.

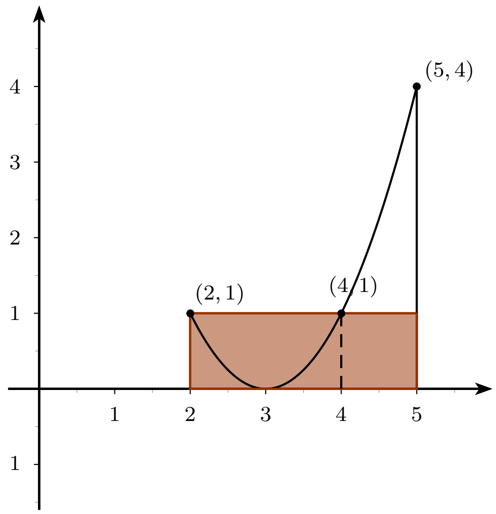
   Solution: 1

   (b) Find c such that $f_{ave} = f(c)$ on $[2, 5]$.

   Solution: 2 and 4

   (c) Sketch the graph of f and a rectangle whose area is the same as the area under the graph of f on $[2, 5]$.

   Solution:



2. Find the volume of the solid generated by revolving the region bounded by $y = \sqrt{x}$, $y = 0$, and $x = 4$ about:

   (a) the $x-$axis

   Solution: $V = \int_0^4 \pi(\sqrt{x})^2\,dx = 8\pi \approx 25.1327$

   (b) the $y-$axis

   Solution: $V = \int_0^2 [4^2 - (y^2)^2]\,dy = \frac{128}{5}\pi \approx 80.42477$

   (c) the line $x = 4$

   Solution: $V = \int_0^2 \pi(4 - y^2)^2\,dy\frac{256}{15}\pi \approx 53.6165$

   (d) the line $x = 6$

   Solution: $V = \int_0^2 \pi[(6 - y^2)^2 - 2^2]\,dy = \frac{192}{5}\pi \approx 120.6372$