Math 520

Arc Length

§6.3

This section uses integrals to calculate the length of a curve (arc length). As with other integral applications, arc length will be the limit of the sums of approximations of the lengths of small pieces of the curve.

Arc Length

If f' is a continuous on $[a, b]$, the length of the curve $y = f(x)$ for $a \leq x \leq b$ is

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$$

For curves described as $x = g(y)$ for $c \leq y \leq d$, the arc length is

$$L = \int_c^d \sqrt{1 + (g'(y))^2}\, dy$$

Distance Along a Curve

For a smooth function $y = f(x)$ for $a \leq x \leq b$ let $s(x)$ be the distance travelled along the curve from $(a, f(a))$ to $(x, f(x))$. The function $s(x)$ may be written as

$$s(x) = \int_a^x \sqrt{1 + (f'(t))^2}\, dt.$$

1. Find the length of $y = \frac{1}{x}$ for $1 \leq x \leq 3$.

Solution: $L = \displaystyle\int_1^3 \sqrt{1 + \left(\frac{-1}{x^2}\right)^2}\, dx \approx 2.14662$

2. Find the arc length of the graph of $y = \sin x$ over one period.

Solution: L= $\displaystyle\int_0^{2\pi} \sqrt{1 + \cos^2 x}\, dx \approx 7.6404$

3. What is the length of the arc given by $x = 1 + y^{3/2}$ for $0 \leq y \leq 4$

Solution: $\displaystyle\int_0^4 \sqrt{1 + \frac{9y}{4}}\, dy \approx 9.07342$

4. Find the length of $y = (4 - x^{2/3})^{3/2}$ for $0 \leq x \leq 2$

Solution: $\displaystyle\int_0^2 \frac{2}{\sqrt[3]{x}}\, dx \approx 4.7622$

5. Find a function for the arc length of the curve $y = 2x^{3/2}$ and starting $(0,0)$.

Solution:

$s(x) = \int_0^x \sqrt{1 + (3t^{1/2})^2}\, dt = \int_0^x \sqrt{1+9t}\, dt.$

Letting $u = 1 + 9t$, then $du = 9dt$. So,

$$\int_0^x \sqrt{1+9t}\, dt = \frac{1}{9}\frac{1+9t^{3/2}}{3/2} = \frac{2}{27}(1+9t)^{3/2}\Big|_0^x = \frac{2}{27}(1+9x)^{3/2} - \frac{2}{27}$$