Math 510

More Tangent and Velocity Problems

§2.6

1. The point $P(\frac{1}{2}, 2)$ lies on the curve $y = \frac{1}{x}$.

(a) If Q is the point $(x, \frac{1}{x})$, use your calculator to find the slope of the secant line PQ (correct to six decimal places) for the following values of x:

(i) 2 (ii) 1 (iii) .9 (iv) .8 (v) .7

(vi) .6 (vii) .55 (viii) .51 (ix) .45 (x) .49

Solution:

	x	Q	m_{PQ}
(i)	2	$(2, .5)$	-1
(ii)	1	$(1, 1)$	-2
(iii)	.9	$(.9, 1.111111)$	-2.222222
(iv)	.8	$(.8, 1.25)$	-2.5
(v)	.7	$(.7, 1.428571)$	-2.857143
(vi)	.6	$(.6, 1.666667)$	-3.333333
(vii)	.55	$(.55, 1.818182)$	-3.636364
(viii)	.51	$(.51, 1.960784)$	-3.921569
(ix)	.45	$(.45, 2.222222)$	-4.444444
(x)	.49	$(.49, 2.040816)$	-4.081633

(b) Using the results of part (a), guess the value of the slope of the tangent line to the curve at $P(\frac{1}{2}, 2)$.

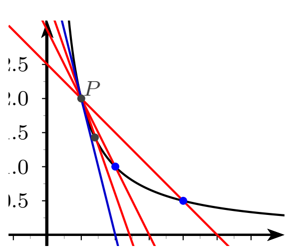
Solution: The numbers for m_{PQ} in the table of part (a) appear to be approaching -4. So, the slope appears to be -4

(c) Using the slope from part (b), find an equation of the tangent line to the curve at $P(\frac{1}{2}, 2)$.

Solution: $y - 2 = -4(x - .5)$ or $y = -4x + 4$

(d) Draw a neat sketch of the curve, two of the secant lines, and the tangent line.

Solution:



2. The displacement (in meters) of a certain particle moving in a straight line is given by $s = t^2 + t$, where t is measured in seconds.

(a) Find the average velocity over the following time periods:

(i) $[0,2]$ (ii) $[0,1]$ (iii) $[0,.5]$ (iv) $[0,.1]$

Solution: The average velocity between times 0 and h is $\dfrac{s(h)-s(0)}{h} = \dfrac{h^2+h-0}{h} = h+1$. So, on
$[0,2]$ $2+1=3$m/s.
$[0,1]$ $1+1=2$m/s.
$[0,.5]$ $.5+1=1.5$m/s.
$[0,.1]$ $.1+1=1.1$m/s.

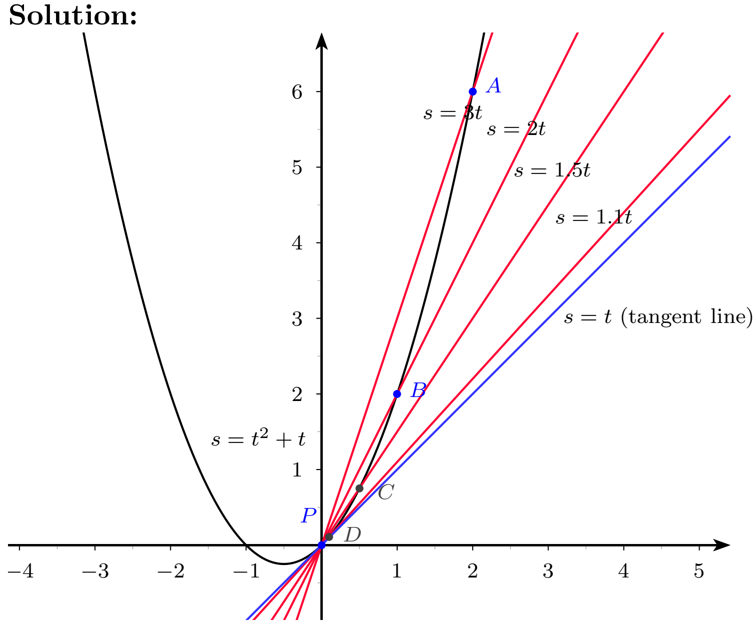
(b) Find the instantaneous velocity when $t=0$.

Solution: The average velocities seem to be approaching 1 m/s. So, I estimate the instantaneous velocity to be 1 m/s.

(c) Draw a neat sketch of the graph of s as a function of t and draw secant lines whose slopes are the average velocities found in part(a).

(d) Draw the tangent line whose slope is the instantaneous velocity from part (b).

Solution:



3. The experimental data in the table below defines y as a function of x.

x	0	1	2	3	4	5
y	2.6	2.0	1.1	1.3	2.1	3.5

(a) If P is the point $(3,1.3)$, find the slopes of the secant lines PQ when Q is the point on the graph with $x=0,1,2,4,5$.

Solution:

x	m_{PQ}
0	$\frac{2.6-1.3}{0-3} \approx -.43$
1	$\frac{2.0-1.3}{1-3} \approx -.35$
2	$\frac{1.1-1.3}{2-3} \approx .2$
4	$\frac{2.1-1.3}{4-3} \approx .8$
5	$\frac{3.5-1.3}{5-3} \approx 1.1$

(b) Estimate the slope of the tangent line at P by averaging the slopes of two appropriate secant lines.

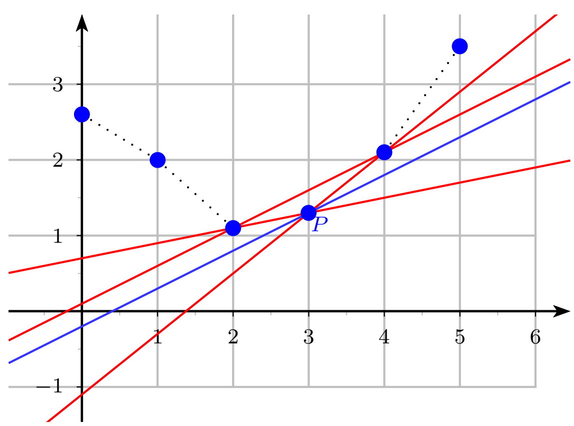
Solution: Use the slope of the secant line between $x = 2$ and $x = 3$ and the slope of the secant line between $x = 3$ and $x = 4$. So we have $\frac{1}{2}(.2 + .8) = .5$.

(c) Determine the slope of the secant line between the points $(2, 1.1)$ and $(4, 2.1)$. How does this compare to your answer for (b)?

Solution: The slope of the secant line is $\frac{2.1-1.1}{4-2} = .5$. This is the same answer as part (b)

(d) Plot the data points. Draw the two secant lines you used in part (b). Draw the secant line from part (c). Draw the tangent line at $x = 3$.

Solution:



Definition The **tangent line** to the curve $y = f(x)$ at point $P(a, f(a))$ is the line through P with slope

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \tag{1}$$

provided the limit exists.

4. Use equation 1, find the equation of the tangent line to the parabola $y = x^2$ at the point $P(1, 1)$.

Solution: Since $a = 1$ and $f(x) = x^2$ we know the slope is

$$\begin{aligned} m &= \lim_{x\to 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x\to 1} \frac{x^2 - 1}{x - 1} \\ &= \lim_{x\to 1} \frac{(x-1)(x+1)}{x-1} \\ &= \lim_{x\to 1} (x+1) = 1 + 1 = 2 \end{aligned}$$

Using the point slope form of the equation of a line, we find that an equation of the tangent line at $(1, 1)$ is

$$y - 1 = 2(x - 1).$$

An equivalent form for the slope of a tangent line can be found by replacing equation 1 with the difference quotient.

Definition The **tangent line** to the curve $y = f(x)$ at point $P(a, f(a))$ is the line through P with slope

$$m = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \tag{2}$$

provided the limit exists.

5. Using equation 2, find an equation of the tangent line to the hyperbola $y = \dfrac{2}{x}$ at the point $(3, 1)$.

Solution:

$$\begin{aligned} m &= \lim_{h\to 0} \frac{f(3+h) - f(3)}{h} \\ &= \lim_{h\to 0} \frac{\frac{3}{3+h} - 1}{h} = \lim_{h\to 0} \frac{\frac{3-(3+h)}{3+h} - 1}{h} \\ &= \lim_{h\to 0} \frac{-h}{h(3+h)} = \lim_{h\to 0} -\frac{1}{3+h} \\ &= \frac{1}{3} \end{aligned}$$

So, an equation of the tangent line at the point $(3, 1)$ is

$$y - 1 = -\frac{1}{3}(x - 3)$$