Math 510

Derivatives 1

§2.7

The derivative of a function $f(x)$ at a point a is

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \tag{1}$$

An equivalent way to write this is

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a} \tag{2}$$

1. Using equation (1), follow these steps to find $f'(3)$ for $f(x) = x^2 + 10x$.

   (a) Find $f(3+h)$.

   Solution: $f(3+h) = (3+h)^2 + 10(3+h) = 9 + 6h + h^2 + 30 + 10h = h^2 - 16h + 39$

   (b) Find $f(3+h) - f(3)$.

   Solution: $f(3+h) - f(3) = h^2 - 16h + 39 - 39 = h^2 - 16h$

   (c) Determine $\dfrac{f(3+h) - f(3)}{h}$ and simplify (if possible) when $h \neq 0$.

   Solution: $\dfrac{f(3+h) - f(3)}{h} = \dfrac{h^2 - 16h}{h} = h - 16$

   (d) Find the limit in part (c).

   Solution: $\lim_{h\to 0} h - 16 = -16$

2. Using equation (2), follow these steps to find $f'(3)$ for $f(x) = 2x - x^2$.

   (a) Find $f(x) - f(3)$.

   Solution: $f(x) - f(3) = 2x - x^2 - (-3) = 2x - x^2 + 3$

   (b) Determine $\dfrac{f(x) - f(3)}{x - 3}$ and simplify (if possible) when $x \neq 3$.

   Solution: $\dfrac{f(x) - f(3)}{x-3} = \dfrac{-x^2 + 2x + 3}{x - 3} = -\dfrac{x^2 - 2x - 3}{x - 3} = -\dfrac{(x-3)(x+1)}{x-3} = -(x+1)$

   (c) Find the limit in part (b).

   Solution: $\lim_{x\to 3} -(x+1) = -4$

3. Follow these steps to find $f'(a)$ when $f(x) = 3 + x^2$.

   (a) Find $f(a+h)$.

   Solution: $f(a+h) = 3 + (a+h)^2 = 3 + a^2 + 2ah + h^2$

(b) Find $f(a+h)-f(a)$.

Solution: $f(a+h)-f(a) = 3+a^2+2ah+h^2-(3+a^2) = 2ah+h^2$

(c) Determine $\dfrac{f(a+h)-f(a)}{h}$ and simplify (if possible) when $h \neq 0$.

Solution: $\dfrac{f(a+h)-f(a)}{h} = \dfrac{2ah+h^2}{h} = 2a+h$

(d) Find the limit in part (c).

Solution: $\lim_{h\to 0} 2a+h = 2a$

The slope m of the tangent line to $f(x)$ at the point a is

$$m = f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \tag{3}$$

An equivalent way to write this is

$$m = f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} \tag{4}$$

Using the point-slope form of the equation a line, you write an equation of the tangent line to f at the point a is

$$y - f(a) = f'(a)(x-a).$$

4. Given that $f'(x) = 3x^2+2$ for $f(x) = x^3+2x+1$, find

(a) the slope of the tangent line to f at the point $x=2$.

Solution: $m = f'(2) = 3(2)^2+2 = 14$

(b) the equation of the line tangent to f at the point $x=2$.

Solution: The point of tangency has x coordinate 2 and y coordinate $f(2) = 2^3+2(2) = 13$. So the equation of the tangent line is $y-13 = 14(x-2)$.

(c) the instantaneous velocity at time $x=4$ if $f(x)$ represents the distance in feet of a particle from the origin at time x.

Solution: $f\prime(4) = 3(4)^2+3 = 51$ft/s.

5. Using equation (3), follow these steps to find the slope and equation of the tangent line to $f(x) = \dfrac{2}{x}$ at $x=1$.

(a) Find $f(1+h)$.

Solution: $f(1+h) = \dfrac{2}{1+h}$

(b) Find $f(1+h)-f(1)$

Solution: $f(1+h) - f(1) = \dfrac{2}{1+h} - \dfrac{2}{1}$

(c) Determine $\dfrac{f(1+h) - f(1)}{h}$ and simplify when $h \neq 0$.

Solution:
$$\frac{f(1+h)-f(1)}{h} = \frac{\frac{2}{1+h} - \frac{2}{1}}{h} = \left(\frac{\frac{2-2(1+h)}{1+h}}{h}\right) = \frac{2-2-2h}{h(1+h)} = \frac{-2h}{h(1+h)} = \frac{-2}{1+h}$$

(d) Find the limit in part (c). This is the slope of the tangent line.

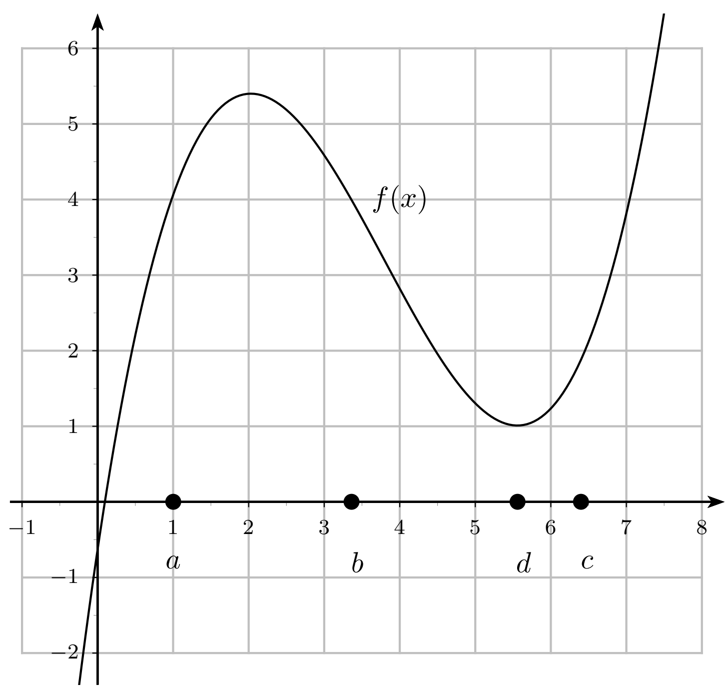
Solution: $\lim\limits_{h \to 0} \dfrac{-2}{1+h} = \dfrac{-2}{1+0} = -2$

(e) Using the point slope form of the equation of a line, $y - f(1) = f'(1)(x-1)$.

Solution: $y - 2 = -2(x-1)$

6. For the graph of the function f given below, arrange the following numbers in increasing order:

$$f'(a) \qquad f'(b) \qquad f'(c) \qquad f'(d)$$



Solution: $f'(b) < f'(d) < f'(a) < f'(c)$