Math 510

Chain Rule

§3.5

This section give another rule for finding derivatives of functions built up from simpler ones. The Chain Rule determines the derivative of a composition of functions. The key to correct use of the Chain Rule comes from first recognizing that the function to be differentiated is a composite and determining the components.

> The Chain Rule: If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $f(g(x))$ is differentiable at x and
>
> $$\frac{d}{dx}f(g(x)) = f'(g(x)) \cdot g'(x)$$
>
> "the derivative of the outer, composed with the inner, times the derivative of the inner"

1. Complete the following

   (a) $f(x) = (x+3)^4$

   outer function(in terms of u): inner function(in terms of x):

   derivative of the outer(du): derivative of the inner(dx):

   Solution:
   outer function(in terms of u): u^4 inner function(in terms of x):$x+3$

   derivative of the outer(du):$4u^3$ derivative of the inner(dx):1

   (b) $f(x) = \cos x^2$

   outer function(in terms of u): inner function(in terms of x):

   derivative of the outer(du): derivative of the inner(dx):

   Solution:
   outer function(in terms of u):$\cos u$ inner function(in terms of x):x^2

   derivative of the outer(du):$-\sin u$ derivative of the inner(dx):$2x$

   (c) $f(x) = \sqrt{1 + \frac{1}{x}}$

   outer function(in terms of u): inner function(in terms of x):

   derivative of the outer(du): derivative of the inner(dx):

   Solution:
   outer function(in terms of u):$\sqrt{u}$ inner function(in terms of x):$1 + x^{-1}$

   derivative of the outer(du):$\frac{1}{2}u^{-\frac{1}{2}}$ derivative of the inner(dx):$-x^{-2}$

2. Find the derivatives of each of the functions in 1 (a)-(c).

Solution:

(a) $f'(x) = 4(x+3)^3$
(b) $f'(x) = (-\sin x^2) \cdot 2x$
(c) $f'(x) = \frac{1}{2}(1+\frac{1}{x})^{-\frac{1}{2}} \cdot (0 + (-1)x^{-2})$

3. Let $h(x) = f(g(x))$, $f'(7) = 3$, $g(4) = 7$, and $g'(4) = 5$. Find $h'(4)$.

Solution: $h'(4) = f'(g(4)) \cdot g'(4) = f'(7) \cdot 5 = 3 \cdot 5 = 15$

4. Determine the derivatives of the following. You do not need to simplify.

(a) $f(x) = \sqrt{x^3 + 6x}$

> **Solution:** $f'(x) = \frac{1}{2}(x^3 + 6x)^{\frac{1}{2}}(3x^2 + 6)$

(b) $f(x) = \sin 2x$

> **Solution:** $f'(x) = 2\cos(2x)$

(c) $f(x) = \sec x^2$

> **Solution:** $f'(x) = (\sec x^2)(\tan x^2)(2x)$

(d) $f(x) = \sin(\sec x)$

> **Solution:** $f'(x) = \cos(\sec x)(\sec x \tan x)$

(e) $f(x) = \sin 2x \cos 3x$

> **Solution:** $f'(x) = (\sin 2x)(-\sin 3x \cdot 3) + (\cos 3x)(\cos 2x \cdot 2)$

(f) $f(x) = \dfrac{1}{(x^2 - 1)^4}$

> **Solution:** Since $f(x) = \dfrac{1}{(x^2 - 1)^4} = (x^2 - 1)^{-4}$ we have $f'(x) = -4(x^2 - 1)^{-5}(2x)$

(g) $f(x) = x^3\sqrt{x^2 + 1}$

> **Solution:** $f'(x) = x^3(\frac{1}{2}(x^2 + 1)^{-\frac{1}{2}} \cdot 2x + (x^2 + 1)^{\frac{1}{2}} \cdot 3x^2$