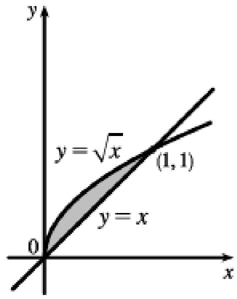
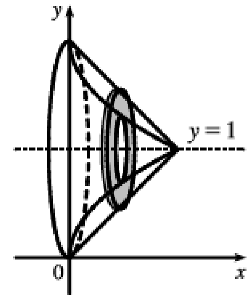
9. A cross-section is a washer with inner radius $1 - \sqrt{x}$ and outer radius $1 - x$, so its area is

$$A(x) = \pi(1-x)^2 - \pi(1-\sqrt{x})^2$$
$$= \pi\left[(1 - 2x + x^2) - (1 - 2\sqrt{x} + x)\right]$$
$$= \pi\left(-3x + x^2 + 2\sqrt{x}\right).$$

$$V = \int_0^1 A(x)\,dx = \pi \int_0^1 \left(-3x + x^2 + 2\sqrt{x}\right) dx$$

$$= \pi\left[-\tfrac{3}{2}x^2 + \tfrac{1}{3}x^3 + \tfrac{4}{3}x^{3/2}\right]_0^1 = \pi\left(-\tfrac{3}{2} + \tfrac{5}{3}\right) = \tfrac{\pi}{6}$$



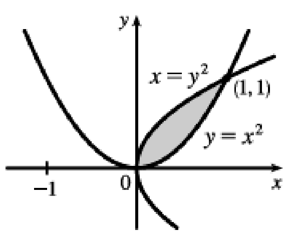


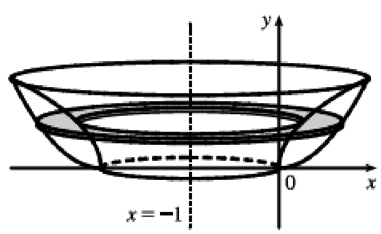
11. $y = x^2 \quad \Rightarrow \quad x = \sqrt{y}$ for $x \geq 0$. The outer radius is the distance from $x = -1$ to $x = \sqrt{y}$ and the inner radius is the distance from $x = -1$ to $x = y^2$.

$$V = \int_0^1 \pi\left\{[\sqrt{y} - (-1)]^2 - \left[y^2 - (-1)\right]^2\right\} dy = \pi \int_0^1 \left[(\sqrt{y} + 1)^2 - \left(y^2 + 1\right)^2\right] dy$$

$$= \pi \int_0^1 \left(y + 2\sqrt{y} + 1 - y^4 - 2y^2 - 1\right) dy = \pi \int_0^1 \left(y + 2\sqrt{y} - y^4 - 2y^2\right) dy$$

$$= \pi\left[\tfrac{1}{2}y^2 + \tfrac{4}{3}y^{3/2} - \tfrac{1}{5}y^5 - \tfrac{2}{3}y^3\right]_0^1 = \pi\left(\tfrac{1}{2} + \tfrac{4}{3} - \tfrac{1}{5} - \tfrac{2}{3}\right) = \tfrac{29}{30}\pi$$

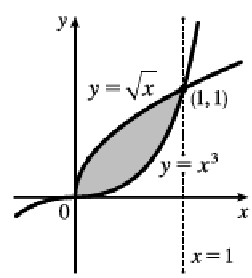




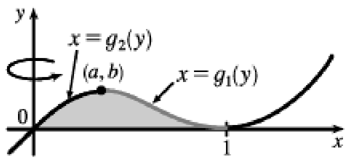
13. $y = \sqrt{x} \quad \Rightarrow \quad x = y^2$ and $y = x^3 \quad \Rightarrow \quad x = \sqrt[3]{y}$. A cross-section is a washer with inner radius $1 - \sqrt[3]{y}$ and outer radius $1 - y^2$, so its area is

$$A(y) = \pi(1-y^2)^2 - \pi\left(1 - \sqrt[3]{y}\right)^2.$$

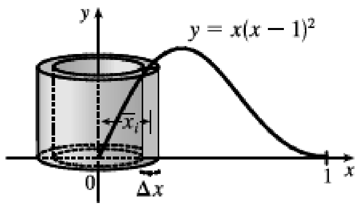
$$V = \int_0^1 A(y)\,dy = \int_0^1 \left[\pi(1-y^2)^2 - \pi\left(1 - \sqrt[3]{y}\right)^2\right] dy$$
$$= \pi \int_0^1 [(1 - 2y^2 + y^4) - (1 - 2y^{1/3} + y^{2/3})]\,dy$$
$$= \pi \int_0^1 (-2y^2 + y^4 + 2y^{1/3} - y^{2/3})\,dy = \pi\left[-\tfrac{2}{3}y^3 + \tfrac{1}{5}y^5 + \tfrac{3}{2}y^{4/3} - \tfrac{3}{5}y^{5/3}\right]_0^1$$
$$= \pi\left(-\tfrac{2}{3} + \tfrac{1}{5} + \tfrac{3}{2} - \tfrac{3}{5}\right) = \tfrac{13\pi}{30}$$



49.



If we were to use the "washer method," we would first have to locate the local maximum point (a, b) of $y = x(x-1)^2$ using the methods of Chapter 4. Then we would have to solve the equation $y = x(x-1)^2$ for x in terms of y to obtain the functions $x = g_1(y)$ and $x = g_2(y)$ shown in the figure above. This step would be difficult because it involves the cubic formula. Finally we would find the volume using $V = \pi \int_0^b \left\{[g_1(y)]^2 - [g_2(y)]^2\right\} dy$.



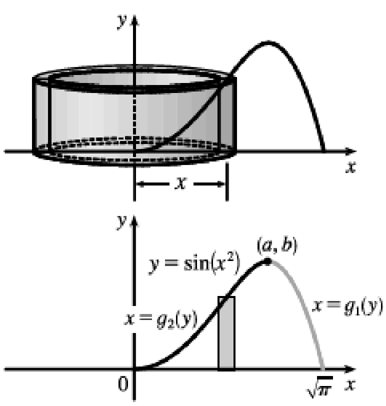
Instead, we use cylindrical shells. As in Example 9, we rotate an approximating rectangle with width Δx about the y-axis, to get a cylindrical shell whose average radius is $\overline{x}_i$ and whose volume is $2\pi\overline{x}_i\left[\overline{x}_i(\overline{x}_i-1)^2\right]\Delta x$. So the total volume is

$$V = \lim_{n\to\infty}\sum_{i=1}^{n} 2\pi\,\overline{x}_i\left[\overline{x}_i(\overline{x}_i-1)^2\right]\Delta x = \int_0^1 2\pi x\left[x(x-1)^2\right]dx$$

$$= 2\pi\int_0^1\left(x^4 - 2x^3 + x^2\right)dx = 2\pi\left[\frac{x^5}{5} - 2\frac{x^4}{4} + \frac{x^3}{3}\right]_0^1$$

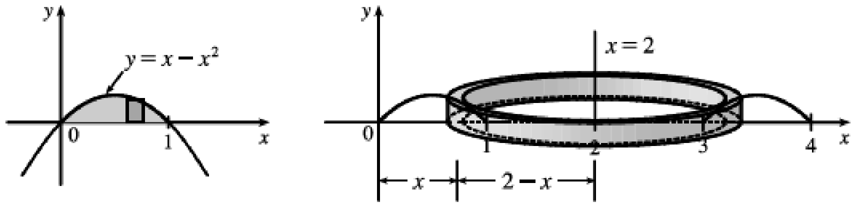
$$= 2\pi\left(\frac{1}{5} - \frac{1}{2} + \frac{1}{3}\right) = 2\pi\left(\frac{1}{30}\right) = \frac{\pi}{15}$$

50.



A typical cylindrical shell has circumference $2\pi x$ and height $\sin(x^2)$.
$V = \int_0^{\sqrt{\pi}} 2\pi x \sin(x^2)\,dx$. Let $u = x^2$. Then $du = 2x\,dx$, so
$V = \pi\int_0^{\pi}\sin u\,du = \pi[-\cos u]_0^{\pi} = \pi[1-(-1)] = 2\pi$.
For slicing, we would first have to locate the local maximum point (a, b) of $y = \sin(x^2)$ using the methods of Chapter 4. Then we would have to solve the equation $y = \sin(x^2)$ for x in terms of y to obtain the functions $x = g_1(y)$ and $x = g_2(y)$ shown in the second figure. Finally we would find the volume using $V = \pi\int_0^b\left\{[g_1(y)]^2 - [g_2(y)]^2\right\}dy$. Using shells is definitely preferable to slicing.

53.

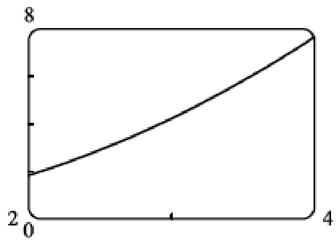


$$V = \int_0^1 (\text{circumference})\,(\text{height})\,(\text{thickness}) = \int_0^1 [2\pi(2-x)]\,(x - x^2)\,dx$$

$$= 2\pi \int_0^1 (x^3 - 3x^2 + 2x)\,dx = 2\pi\left[\tfrac{1}{4}x^4 - x^3 + x^2\right]_0^1 = 2\pi\left(\tfrac{1}{4}\right) = \tfrac{\pi}{2}$$

See the solution for Exercise 49 as to why the method of cylindrical shells is preferable to slicing.

8.



$$y = \frac{x^2}{2} - \frac{\ln x}{4} \quad \Rightarrow \quad \frac{dy}{dx} = x - \frac{1}{4x} \quad \Rightarrow$$

$$1 + \left(\frac{dy}{dx}\right)^2 = x^2 + \frac{1}{2} + \frac{1}{16x^2}. \text{ So}$$

$$L = \int_2^4 \left(x + \frac{1}{4x}\right) dx = \left[\frac{x^2}{2} + \frac{\ln x}{4}\right]_2^4 = \left(8 + \frac{2\ln 2}{4}\right) - \left(2 + \frac{\ln 2}{4}\right)$$

$$= 6 + \frac{\ln 2}{4}$$

7. (a) $f_{\text{ave}} = \dfrac{1}{\pi - 0}\displaystyle\int_0^\pi (2\sin x - \sin 2x)\,dx$

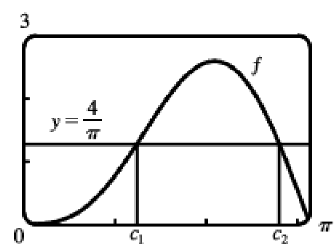
$$= \tfrac{1}{\pi}\left[-2\cos x + \tfrac{1}{2}\cos 2x\right]_0^\pi$$

$$= \tfrac{1}{\pi}\left[\left(2 + \tfrac{1}{2}\right) - \left(-2 + \tfrac{1}{2}\right)\right] = \tfrac{4}{\pi}$$

(b) $f(c) = f_{\text{ave}} \quad \Leftrightarrow \quad 2\sin c - \sin 2c = \frac{4}{\pi} \quad \Leftrightarrow$

$c_1 \approx 1.238$ or $c_2 \approx 2.808$

(c)



8. (a) $f_{\text{ave}} = \dfrac{1}{2 - 0}\displaystyle\int_0^2 \frac{2x}{(1+x^2)^2}\,dx$

$$= \frac{1}{2}\int_1^5 \frac{1}{u^2}\,du \qquad [u = 1 + x^2,\, du = 2x\,dx]$$

$$= \frac{1}{2}\left[-\frac{1}{u}\right]_1^5 = -\frac{1}{2}\left(\frac{1}{5} - 1\right) = \frac{2}{5}$$

(b) $f(c) = f_{\text{ave}} \quad \Leftrightarrow \quad \dfrac{2c}{(1+c^2)^2} = \dfrac{2}{5} \quad \Leftrightarrow \quad 5c = (1 + c^2)^2 \quad \Leftrightarrow$

$c_1 \approx 0.220$ or $c_2 \approx 1.207$

(c)